γ_π-factors for principal series & special reps

Recall the setting and result of last time:

$G = GL_2(F)$ $\quad$ F p-adic field $\quad$ $\tau : F \to \mathbb{C}^\times$

π irreducible admissible rep of G

Kirillov model

$$\mathcal{K}(\pi) = \left\{ \begin{array}{l} \xi : F^\times \to \mathbb{C} \text{ locally constant}, \\ \pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \xi(x) = \tau(bx)\xi(ax) \end{array} \right\}$$

For $\xi \in \mathcal{K}(\pi)$ and $\chi : F^\times \to \mathbb{C}^\times$ character, set

$$M_\xi(\chi, s) = \int_{F^\times} \xi(x)\, \chi(x)^{-1}\, |x|^{2s-1}\, d^\times x \qquad s \in \mathbb{C}$$

<u>Theorem</u> (equivalent to [Godement, Thm 8])

(a) $M_\xi(\chi, s)$ converges for $\mathrm{Re}(s) \gg 0$

(b) $M_\xi(\chi, s)$ can be analytically continued to a meromorphic function with ≤ 2 poles.

(c) $\exists$ meromorphic $\gamma_\pi(\chi, s)$ satisfying

$$M_{\pi(\omega)\xi}(\omega_\pi - \chi, 1-s) = \gamma_\pi(\chi, s) M_\xi(\chi, s)$$

and

$$\gamma_\pi(\omega_\pi - \chi, 1-s)\, \gamma_\pi(\chi, s) = \omega_\pi(-1)$$

where $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G$ and ω_π is the central character of π.

Two aims for today:

① Describe the corresponding result for $GL_1(F)$

② For $\pi = \pi_{\mu_1, \mu_2}$ in the principal series or a special representation, express the factors γ_π appearing for $GL_2(F)$ in terms of the analogous factors appearing for $GL_1(F)$.

Tate's thesis: p-adic GL_1

For $\varphi \in \mathcal{S}(F)$ and $\chi : F^\times \to \mathbb{C}^\times$ character, set

$$L_\varphi(\chi, s) = \int_{F^\times} \varphi(x)\chi(x)|x|^s d^\times x \qquad (166)$$

Theorem (Tate) (see e.g. Thm 4.18 in Poonen's notes)

(a) $L_\varphi(\chi,s)$ converges for $\mathrm{Re}(s) \gg 0$ (if χ unitary, for $\mathrm{Re}(s) > 0$).

(b) $L_\varphi(\chi,s)$ has meromorphic continuation to $\mathbb{C}$

(c) $\exists\, \gamma(\chi,s)$ such that

$$L_\varphi(-\chi, 1-s) = \gamma(\chi,s) L_{\hat{\varphi}}(\chi,s) \qquad (167)$$

and $\gamma(\chi,s)\gamma(-\chi,1-s) = \chi(-1)$, $\qquad (180)$

where $\hat{\varphi}$ is the Fourier transform of φ.

There is more to be said. Define

$$L(\chi,s) = \begin{cases} 1 & \text{if } \chi \text{ ramified} \\ \left(1-\chi(\varpi)q^{-s}\right)^{-1} & \text{if } \chi \text{ unramified} \end{cases} \qquad (187)$$

Fix χ.

For every $\varphi \in \mathcal{S}(F)$, $\dfrac{L\varphi(\chi,s)}{L(\chi,s)}$ is entire in s.

There exists $\varphi \in \mathcal{S}(F)$ s.t. $\dfrac{L\varphi(\chi,s)}{L(\chi,s)} = 1$.

We say that $L(\chi,s)$ is the gcd of $\{L\varphi(\chi,s) \mid \varphi \in \mathcal{S}(F)\}$.

This has two consequences for us:

- part (b) of the Theorem can be made more precise:
  - if χ is ramified, then $L\varphi(\chi,s)$ is entire
  - if χ is unramified, then $L\varphi(\chi,s)$ has at most a single, simple pole.

- we can use $L(\chi,s)$ as a "correction factor" and consider the quotient $\dfrac{L\varphi(\chi,s)}{L(\chi,s)}$; then the functional equation can be written as

$$\frac{L\varphi(-\chi,1-s)}{L(-\chi,1-s)} = \varepsilon(\chi,s)\frac{L\hat{\varphi}(\chi,s)}{L(\chi,s)}$$

For fixed χ, $\varepsilon(\chi,s) = Ae^{Bs}$ for some $A,B \in \mathbb{C}$ (depending on the additive char τ and the measure dx on F)

A furtive global glimpse:
Hecke characters and Hecke L-functions

Let K be a number field, e.g. $K = \mathbb{Q}$.
A Hecke character of K is a continuous character

$$\chi : GL_1(\mathbb{A}_K) / GL_1(K^\times) \longrightarrow \mathbb{C}^\times$$

where $GL_1(\mathbb{A}_K)$ is the idele group of K.
By the definition of the idele topology,

$$\chi = \prod_v \chi_v \qquad (v \text{ places of } K)$$

where $\chi_v : K_v^\times \longrightarrow \mathbb{C}^\times$ is unramified for all but finitely many non-archimedean places v of K.
The (global) L-function of χ is

$$L(\chi, s) = \prod_{\substack{\mathfrak{p}_v \subset \mathcal{O}_K \\ \text{s.t. } \chi_v \text{ unram.}}} \left(1 - \frac{\chi_v(\mathfrak{p}_v)}{N(\mathfrak{p}_v)^s}\right)^{-1}$$

By global class field theory, this Hecke L-function agrees with the Artin L-function of the 1-dimensional Galois representation corresponding to χ. The Euler factors of this Artin L-function are defined using the characteristic polynomials of Frobenius elements.

For an interesting discussion of local zeta integrals themselves, see the introduction to Wen-Wei Li's book "Zeta integrals, Schwartz spaces and local functional equations" LNM 2228.

Back to GL_2 now.

Outline ([Godement, Section 13])

- take $\xi \in \mathcal{S}(F^\times) \cap \pi(w)\mathcal{S}(F^\times)$ such that

  $M_\xi(\chi, s) \not\equiv 0$.

- show that $M_\xi(\chi, s) = L_{\xi'}(\mu_2 - \chi, \overbrace{2s - \tfrac{1}{2}}^{s'})$ for suitable ξ'

  and deduce

$$\gamma(\mu_2 - \chi, s') M_\xi(\chi, s) = \tag{176}$$

$$= \int_{F^\times} \varphi\left(w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)\chi(y)\mu_2^{-1}(y)|y|^{1-s'} d^\times y$$

for suitable φ

- similarly,

$$M_{\pi(w)\xi}(\omega_\pi - \chi, 1-s) = L_{\eta'}(\chi - \mu_1, 1-s') \tag{177}$$

and

$$\gamma(\chi - \mu_1, s') M_{\pi(w)\xi}(\omega_\pi - \chi, 1-s) = \tag{178}$$

$$= \mu_1(-1)\chi^{-1}(-1) \int_{F^\times} \varphi\left(w^{-1}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)\chi(y)\mu_2^{-1}(y)|y|^{1-s'} d^\times y$$

Putting these together,

$$\gamma_\pi(\chi,s) = \frac{M_{\pi(\omega)\xi}(\omega_\pi - \chi, 1-s)}{M_\xi(\chi,s)} \tag{179}$$

$$= \mu_1(-1)\chi^{-1}(-1)\,\frac{\gamma(\mu_2-\chi,s')}{\gamma(\chi-\mu_1,1-s')}$$

But $\gamma(\chi,s)\gamma(-\chi,1-s) = \chi(-1)$

applied to $(\mu_1-\chi)$ gives

$$\gamma_\pi(\chi,s) = \gamma(\mu_1-\chi, 2s-\tfrac{1}{2})\,\gamma(\mu_2-\chi,2s-\tfrac{1}{2})$$

This is [Godement, Theorem 9].

We delve into the details now.
Recall that $\pi = \pi_{\mu_1,\mu_2}$ and ξ is chosen so that

$$\xi \in \mathcal{S}(F^\times) \cap \pi(\omega)\mathcal{S}(F^\times) \tag{168}$$

and $M_\xi(\chi,s) \not\equiv 0$.

Consider $\Phi : F \to \mathbb{C}$ defined by

$$\Phi(y) = \int_F \mu_2^{-1}(x)|x|^{-1/2}\xi(x)\tau(xy)\,dx$$

It's the Fourier transform of an element of $\mathcal{S}(F^\times)$, so $\Phi \in \mathcal{S}(F) \subset \mathcal{F}_\mu$, where

$$\mathcal{F}_\mu = \left\{ \Phi : F \to \mathbb{C} \text{ loc. const} \;\middle|\; \Phi(y)\mu(y)|y| \text{ constant for } |y| \text{ large} \right\}$$

Recall that

$$\mathcal{B}_{\mu_1,\mu_2} = \left\{ \varphi : G_F \to \mathbb{C} \text{ loc. const} \;\; \varphi\left(\begin{bmatrix} t_1 & * \\ 0 & t_2 \end{bmatrix} g\right) = \mu_1(t_1)\mu_2(t_2)\left|\frac{t_1}{t_2}\right|^{1/2}\varphi(g) \right\}$$

$$\begin{aligned} \mathcal{F}_\mu &\xrightarrow{\;\cong\;} \mathcal{B}_{\mu_1,\mu_2} \\ \Phi &\longmapsto \varphi(g) = \mu_1(\det g)|\det g|^{1/2}\mu^{-1}(c)|c|^{-1}\Phi\left(\frac{d}{c}\right) \end{aligned}$$

for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $c \neq 0$

$$\Phi(y) = \varphi\left(w^{-1}\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right) \longleftarrow\!\shortmid \varphi$$

So we have $\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ with

$$\varphi\left(w^{-1}\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}\right) = \int_F \mu_2^{-1}(x)\,|x|^{-1/2}\,\xi(x)\,\tau(xy)\,dx \tag{169}$$

Fourier inversion gives

$$\xi(x) = \mu_2(x)\,|x|^{1/2}\int_F \varphi\left(w^{-1}\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}\right)\overline{\tau}(xy)\,dy$$

$$= \xi_\varphi(x) \tag{170}$$

where ([Godement, Section 10])

$$\mathcal{B}_{\mu_1,\mu_2} \hookrightarrow \mathcal{K}(\pi) \qquad \text{if } \mu(x) \neq |x|^{-1}$$

$$\varphi \longmapsto \xi_\varphi \qquad \left(\begin{array}{l}\text{this holds wlog}\\ \text{as } \pi_{\mu_2,\mu_1} = \pi_{\mu_1,\mu_2}\end{array}\right)$$

This embedding is G_F-equivariant, so

$$\big(g \mapsto \varphi(gw)\big) \longmapsto \pi(w)\xi = \pi(w)\xi_\varphi$$

But we chose ξ so that $\pi(w)\xi \in \mathcal{S}(F^\times)$, so we can repeat the calculations with ξ replaced by $\pi(w)\xi$ and φ replaced by $g \mapsto \varphi(gw)$ to get

$$\varphi\left(w^{-1}\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}w\right) = \int_F \mu_2^{-1}(x)|x|^{-1/2}\pi(w)\xi(x)\tau(xy)\,dx \tag{171}$$

We work on the LHS of this:

$$\varphi\left(w^{-1}\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}w\right) = \varphi\left(\begin{bmatrix}1 & -y^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}y^{-1} & 0\\ 0 & y\end{bmatrix} w \begin{bmatrix}1 & -y^{-1}\\ 0 & 1\end{bmatrix}\right)$$

$(\varphi \in \mathcal{B}_{\mu_1,\mu_2})$

$$= \mu_1(y^{-1})\mu_2(y)\left|\frac{y^{-1}}{y}\right|^{1/2}\varphi\left(\underset{\substack{\| \\ -w^{-1}}}{w}\begin{bmatrix}1 & -y^{-1}\\ 0 & 1\end{bmatrix}\right)$$

$(\mu = \mu_1 - \mu_2)$

$$= \omega_\pi(-1)\mu(y^{-1})|y^{-1}|\varphi\left(w^{-1}\begin{bmatrix}1 & -y^{-1}\\ 0 & 1\end{bmatrix}\right) \tag{173}$$

Fourier inversion gives

$$\eta'(x) := \mu_2^{-1}(x)|x|^{-1/2}\pi(w)\xi(x) = \tag{174}$$

$$= \omega_\pi(-1)\int_F \mu(y^{-1})|y^{-1}|\varphi\left(w^{-1}\begin{bmatrix}1 & -y^{-1}\\ 0 & 1\end{bmatrix}\right)\overline{\tau}(xy)\,dy$$

Also set

$$\xi'(x) := \mu_2^{-1}(x)|x|^{-1/2}\xi(x) = \tag{175}$$

$$= \int_F \varphi\left(w^{-1}\begin{bmatrix}1 & -y^{-1}\\ 0 & 1\end{bmatrix}\right)\overline{\tau}(xy)\,dy$$

Finally getting to the local zeta integrals:

$$M_{\xi}(\chi, s) = \int_{F^\times} \xi(x)\, \chi(x)^{-1} |x|^{2s-1} d^\times x$$

$$= \int_{F^\times} \xi'(x)\, \mu_2(x)\, \chi^{-1}(x)\, |x|^{2s-\frac{1}{2}} d^\times x$$

$$= L_{\xi'}(\mu_2 - \chi, s')$$

Setting $\xi''(y) = \varphi\left(w^{-1}\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right)$, we have $\xi' = \widehat{\xi''}$ so the $GL_1(F)$ functional equation gives

$$\gamma(\mu_2 - \chi, s') M_{\xi}(\chi, s) = \gamma(\mu_2 - \chi, s') L_{\widehat{\xi''}}(\mu_2 - \chi, s')$$

$$= L_{\xi''}(\chi - \mu_2, 1 - s') \tag{176}$$

$$= \int \varphi\left(w^{-1}\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right) \chi(y)\, \mu_2^{-1}(y)\, |y|^{1-s'} d^\times y$$

and similarly for

$\gamma(\chi - \mu_1, s') M_{\pi(w)\xi}(\omega_\pi - \chi, 1 - s)$.